Aggregated kernel based tests in a regression model

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Résumé. Dans un modéle de régression $Y_i = f(X_i)+\sigma\epsilon_i,\ i = 1, \cdots, n$, nous abordons la question du test de la nullité de la fonction f. Nous proposons tout d'abord une nouvelle procédure de test unique basée sur un noyau symétrique général et une estimation de la variance des observations. Les valeurs critiques correspondantes sont construites pour obtenir des tests de niveau non asymptotiques α. Nous introduisons ensuite une procédure d'agrégation afin d'optimism le choix des parametres du noyau. Les tests multiples vérifient les propriétés non asymptotiques et adaptatives au sens minimax de plusieurs classes dalternatives classiques.

Mots-clés. Taux de séparation, tests adaptatifs, modle de rgression, Les méthodes basées sur l'utilisation de noyaux, tests nationaux.

Abstract. Considering a regression model $Y_i = f(X_i)+\sigma\epsilon_i,\ i = 1, \cdots, n$, we address the question of testing the nullity of the function f. The testing procedure is available when the variance of the observations is unknown and does not depend on any prior information on the alternative. We first propose a single testing procedure based on a general symmetric kernel and an estimation of the variance of the observations. The corresponding critical values are constructed to obtain non asymptotic level-α tests. We then introduce an aggregation procedure to avoid the difficult choice of the kernel and of the parameters of the kernel. The multiple tests satisfy non-asymptotic properties and adaptive in the minimax sense over several classes of regular alternatives.

Keywords. Separation rates, adaptive tests, regression model, kernel methods, aggregated test.

1 Introduction

We consider the regression model $Y_i = f(X_i)+\sigma\epsilon_i,\ i = 1, \cdots, n$, where $X = (X_1, X_2, \cdots, X_n)$ are random variables observed in a measurable space E in $\mathbb{R}$ and $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ are i.i.d standard Gaussian variables, independent of $(X_1, X_2, \cdots, X_n)$. Let ν be a measure on E, f is assumed to be in $\mathbb{L}^2(E, d\nu)$ and σ is unknown. In order to be able to estimate σ^2, we assume that we also observe $\left(Y_1^{'}, \cdots, Y_n^{'}\right)$ that obey to the above model $Y_i^{'} = f\left(\frac{i}{n}\right) + \sigma\epsilon_i^{'}, i = 1, \cdots, n$, with $\epsilon^{'} = \left(\epsilon_1^{'}, \cdots, \epsilon_n^{'}\right)$ is independence of $(Y_1, \cdots, Y_n)$. Given the observation of $(X_i, Y_i)_{1\leq i\leq n}$, $\left(Y_i^{'}\right)_{1\leq i\leq n}$, we address the question of testing the null hypothesis $(H_0):\ f = 0$, and the alternative $(H_1):\ f \neq 0$.

In our work, we propose to construct aggregated kernel based testing procedures of (H_0) versus (H_1) in a regression model. Our test statistics are based on a single kernel function which can chosen either as a projection or Gaussian kernel and we propose an estimation for the unknown variance σ^2. Our tests are exactly (and not only asymptotically) of level α. We obtain the optimal non-asymptotic conditions on the alternative which guarantee that the probability of second kind error is at most equal to precribed level β. However, the testing procedures that we introduce hereafter also intended to overcome the question of calibrating the choice of kernel and/or the parameters of the kernel. This is based on an aggregation approach, that is well-known in adaptive testing (Baraud, Huet and Laurent (2003); Fromont, Laurent and Reynaud-Bouret (2013)). This paper is strengly inspired by the paper of Fromont, Laurent and Reynaud-Bouret (2013). Instead of considering a particular single kernel, we consider a collection of kernels and the corresponding collection of tests, each with an adapted level of significance. We then reject the null hypothesis when there exists at least one of the tests in the collection which rejects the null hypothesises. The multiple testing procedures are constructed to be of level α and the loss in second kind error due to the aggregation, when unavoidable, is as small as possible. At last, we compare our tests with tests investigated in Eubank and LaRiccia (1993) from a practical point of view.

2 Single tests based on a single kernel.

2.1 Definition of the testing procedure.

Let K be any symmetric kernel function: $E \times E \to \mathbb{R}$ satisfying: $\int_{E^2} K^2(x,y) f(x) f(y) d\nu(x) d\nu(y) < +\infty$. We introduce the test statistic V_K defined as follows,

$$V_K = \frac{T_K}{\hat{\sigma}_n^2} = \frac{\frac{1}{n(n-1)} \sum_{i \neq j=1}^{n} K(X_i, X_j) Y_i Y_j}{\frac{1}{n} \sum_{i=1}^{n/2} \left(Y'_{2i-1} - Y'_{2i} \right)^2}. \tag{1}$$

We denote for all $x \in E$, $K[f](x) = \int_E K(x,y) f(y) d\nu(y)$, and for all $f, g \in \mathbb{L}^2(E, d\nu)$, $\langle f, g \rangle = \int_E f(x) g(x) d\nu(x)$ associated with $\|f\|^2 = \langle f, f \rangle$.
Thus $\mathbb{E}(T_K) = \langle K[f], f \rangle$, whose existence is ensured by the above aussumtion of K. On the other hand, we have $\widehat{\sigma}_n^2$ is a biased estimator of σ^2 with bias $a^2 := \frac{1}{n} \sum_{i=1}^{n/2} \left[f\left(\frac{2i-1}{n}\right) - f\left(\frac{2i}{n}\right) \right]^2$. If f is a regular function this bias will be small. V_K is a proposed test of $f = 0$ since $\mathbb{E}(T_K)$ which can be viewed as an estimate of $\|f\|^2$. To see this, we consider two following examples.
Example 1. $E = [0,1]$, K is a symmetric kernel function based on a finite orthonormal family $\{\phi_\lambda,\ \lambda \in \Lambda\}$ with respect to the scalar product $\langle ., . \rangle$, $K(x,y) = \sum_{\lambda \in \Lambda} \phi_\lambda(x) \phi_\lambda(y)$. We have $\mathbb{E}(T_K) = \langle \Pi_S(f), f \rangle$, where S is the subspace of $\mathbb{L}^2([0,1], d\nu)$ generated by $\{\phi_\lambda,\ \lambda \in \Lambda\}$ and Π_S denotes the orthogonal projection onto S for $\langle ., . \rangle$. Hence, when $\{\phi_\lambda,\ \lambda \in \Lambda\}$ is well-chosen, T_K can also be viewed as a relevant estimator of $\|f\|^2$.
Example 2. When $E = \mathbb{R}$ and ν is a density function respect to the Lebesgue measure on $\mathbb{R}$, K is a Gaussian kernel, $K(x,y) = \frac{1}{h} k\left(\frac{x-y}{h}\right), \forall (x,y) \in \mathbb{R}^2$, with $k(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-u^2/2\right), \forall\, u \in \mathbb{R}$ and h is a positive bandwidth. We have $\mathbb{E}(T_K) = \langle k_h * f, f \rangle$, where $*$ is the usual convolution operator with respect to the measure ν and $k_h(u) = \frac{1}{h} k\left(\frac{u}{h}\right),\ \forall u \in \mathbb{R}$. Hence, when the bandwidth h is well chosen, T_K can also be viewed as a relevant estimator of $\|f\|^2$.
Hence it is reasonable proposal to consider a test which rejects (H_0) when V_K is as "large enough".

Now, we define critical values used in our tests within the meaning of "large enough". We denote

$$V_K^{(0)} = \frac{\frac{1}{n(n-1)} \sum_{i \neq j=1}^{n} K(X_i, X_j) \epsilon_i \epsilon_j}{\frac{1}{n} \sum_{i=1}^{n/2} \left(\epsilon'_{2i-1} - \epsilon'_{2i} \right)^2}, \tag{2}$$

under (H_0), conditionally on X, V_K and $V_K^{(0)}$ have exactly the same distribution. For $\alpha \in (0,1)$, we now choose the quantile $q_{K,1-\alpha}^{(X)}$ which can be approximated by a classical Monte Carlo method, of the conditional distribution of $V_K^{(0)}$ given X as critical value for our test. We consider the test that rejects (H_0) when $V_K > q_{K,1-\alpha}^{(X)}$, $\Phi_{K,\alpha} = \mathbb{1}\{V_K > q_{K,1-\alpha}^{(X)}\}$.

2.2 Probabilities of first and second kind errors of the test.

Under (H_0), we see that V_K and $V_K^{(0)}$ have exactly the same distribution conditionally on X. As a result, given $\alpha \in (0,1)$, under (H_0) and by taking the expectation over X, we obtain $\mathbb{P}_{(H_0)}(\Phi_{K,\alpha} = 1) \leq \alpha$.
For alternative hypothesis, given β in $(0,1)$, we denote by $q_{K,1-\beta/2}^{\alpha}$ the $(1-\beta/2)$ quantile of the conditional quantile $q_{K,1-\alpha}^{(X)}$, so $\mathbb{P}_f(\Phi_{K,\alpha} = 0) \leq \mathbb{P}_f\left(V_K \leq q_{K,1-\beta/2}^{\alpha}\right) + \beta/2$. The following proposition gives a condition which ensures that $\mathbb{P}_f(\Phi_{K,\alpha} = 0) \leq \beta$.

Proposition 2.1. *Let $\alpha,\ \beta \in (0,1)$. We have that $\mathbb{P}_f\left(V_K \leq q_{K,1-\beta/2}^{\alpha}\right) \leq \beta/2$, as soon as $\langle K[f], f\rangle \geq \sqrt{\frac{16A_K+8B_K}{\beta}} + D_{n,\beta}\ q_{K,1-\beta/2}^{\alpha}$, for $A_K = \frac{n-2}{n(n-1)}\int_E (K[f](x))^2\left[f^2(x)+\sigma^2\right]d\nu(x)$, $D_{n,\beta} = \sigma^2 + a^2 + \frac{4\sigma^2}{n}\sqrt{\left(\frac{n}{2}+\frac{na^2}{\sigma^2}\right)\ln\left(\frac{2}{\beta}\right)} + \frac{4\sigma^2}{n}\ln\left(\frac{2}{\beta}\right)$ and $B_K = \frac{1}{n(n-1)}\int_{E^2} K^2(x,y)\left[f^2(x)+\sigma^2\right]\left[f^2(y)+\sigma^2\right]d\nu(x)d\nu(y)$. Thus under condition of $\langle K[f], f\rangle$, we have $\mathbb{P}_f(\Phi_{K,\alpha} = 0) \leq \beta$. Moreover, there exists some constant $\kappa > 0$ such that, for every K and $n \geq 32\ln(2/\alpha)$, $q_{K,1-\beta/2}^{\alpha} \leq \frac{2\kappa}{\sqrt{n(n-1)}}\ln\left(\frac{2}{\alpha}\right)\sqrt{\frac{2\int_{E^2}K^2(x,y)d\nu(x)d\nu(y)}{\beta}}$.*

The following theorem gives a condition on $\|f\|^2$ for the test to be powerful.

Theorem 2.2. *Let $\alpha,\ \beta \in (0,1)$, κ be a positive constant, K be a symmetric kernel function, C_K be an upper bound for $\int_{E^2} K^2(x,y)d\nu(x)d\nu(y)$. Then $\forall n \geq 32\ln(2/\alpha)$, we have $\mathbb{P}_f(\Phi_{K,\alpha} = 0) \leq \beta$, as soon as $\|f\|^2 \geq \|f - K[f]\|^2 + \frac{64(n-2)\left(\|f\|_\infty^2+\sigma^2\right)}{n(n-1)\beta} + \frac{4}{\sqrt{n(n-1)\beta}}\left(\kappa D_{n,\beta}\ln\left(\frac{2}{\alpha}\right) + 4\left(\|f\|_\infty^2+\sigma^2\right)\right)\sqrt{C_K}$, where for any real valued function f, $\|f\|_\infty = \sup_{x\in E}|f(x)|$.*

2.3 Performance of the Monte Carlo approximation.

In this section, we introduce a Monte Carlo method used to approximate the conditional quantiles $q_{K,1-\alpha}^{(X)}$ by $\hat{q}_{K,1-\alpha}^{(X)}$. Conditionally on X, we consider $\{\epsilon^b,\ 1 \leq b \leq B\}$ and $\{\epsilon'^b,\ 1 \leq b \leq B\}$, with $\epsilon^b = \{\epsilon_i^b\}_{i=1}^n$, $\epsilon'^b = \{\epsilon_i'^b\}_{i=1}^n$ are sequences of i.i.d standard Gaussian random variables and $\{\epsilon^b,\ 1 \leq b \leq B\}$, $\{\epsilon'^b,\ 1 \leq b \leq B\}$ are assumed to be independent. Under conditionally on X, for $1 \leq b \leq B$, we define

$$V_K^{\left(\epsilon^b,\epsilon'^b\right)} = \frac{\frac{1}{n(n-1)}\sum_{i\neq j=1}^n K(X_i,X_j)\epsilon_i^b\epsilon_j^b}{\frac{1}{n}\sum_{i=1}^{n/2}\left(\epsilon_{2i-1}'^b - \epsilon_{2i}'^b\right)^2}.$$

Under (H_0), conditionally on X, the variables $V_K^{\left(\epsilon^b,\epsilon'^b\right)}$ have the same distribution function as V_K and $V_K^{(0)}$. Denoting $F_{K,B} = \frac{1}{B}\sum_{b=1}^B \mathbb{1}\left\{V_K^{\left(\epsilon^b,\epsilon'^b\right)} \leq x\right\}$, $\forall x \in \mathbb{R}$. Then

$\hat{q}^{(X)}_{K,1-\alpha} = F^{-1}_{K,B}(\alpha) = \inf\{t \in \mathbb{R},\ F_{K,B}(t) \geq 1-\alpha\}$ and we now consider the test $\widehat{\Phi}_{K,\alpha} = \mathbb{1}\left\{V_K > \hat{q}^{(X)}_{K,1-\alpha}\right\}$. Approaching this proposed test, the probabilities of first and second kind errors of the test built with these Monte Carlo approximation are guaranteed from results in paper of Fromont, Laurent and Reynaud-Bouret (2013).

3 Multiple tests based on collections of kernel functions.

Following this section, we consider some collections of kernel functions instead of a single one to avoid choosing the kernel, and/or its parameters when considering the testing procedures based on a single kernel function K.. Introducing a finite collection $\{K_m,\ m \in \mathcal{M}\}$ of symmetric kernel functions: $E \times E \to \mathbb{R}$. For $m \in \mathcal{M}$,we define V_{K_m} and $V^{(0)}_{K_m}$ corresponding to K_m replaced in (1) and (2) and $\{w_m,\ m \in \mathcal{M}\}$ be a collection of positive numbers such that $\sum_{m\in\mathcal{M}} e^{-w_m} \leq 1$. Conditionally on X, for $u \in (0,1)$, we denote by $q^{(X)}_{m,1-u}$ the $(1-u)$ quantile of $V^{(0)}_{K_m}$. Given α in $(0,1)$, we consider the test which rejects (H_0) when there exists at least one m in $\mathcal{M}$ such that $V_{K_m} > q^{(X)}_{m,1-u^{(X)}_\alpha e^{-w_m}}$, where $u^{(X)}_\alpha = \sup\left\{u > 0,\ \mathbb{P}\left(\sup_{m\in\mathcal{M}}\left(V_{K_m} - q^{(X)}_{m,1-ue^{-w_m}}\right) > 0 \middle| X\right) \leq \alpha\right\}$. We consider the corresponding test function $\Phi_\alpha = \mathbb{1}\left\{\sup_{m\in\mathcal{M}}\left(V_{K_m} - q^{(X)}_{m,1-u^{(X)}_\alpha e^{-w_m}}\right) > 0\right\}$.
Using the Monter Carlo method, we can estimate $u^{(X)}_\alpha$ and the quantile $q^{(X)}_{m,1-u^{(X)}_\alpha}$ by given X. The level $\alpha \in (0,1)$ and the probability of second kind error at most equal to $\beta \in (0,1)$ can be guaranteed for one of the single tests rejecting (H_0) when $V_{K_m} > q^{(X)}_{m,1-u^{(X)}_\alpha e^{-w_m}}$.

3.1 Presentation of the simulation study.

We particularly study our multiple testing procedures in this section with $E = [0,1]$, $n = 100$ and $\alpha = 0.05$, X is a uniform variable on $[0,1]$. First, we consider the Haar basis $\{\phi_0,\ \phi_{(j,k)},\ j \in \mathbb{N}, k \in \{0,\cdots,2^j-1\}$, $\phi_0(x) = \mathbb{1}[0,1](x)$ and $\phi_{(j,k)}(x) = 2^{j/2}\psi(2^jx-k)$, with $\psi(x) = \mathbb{1}\,[0,1/2)\,(x) - \mathbb{1}\,[1/2,1]\,(x)$. Let $K_0(x,x^{'}) = \phi_0(x)\phi_0(x^{'})$ and for $J \geq 1$ $K_J(x,x^{'}) = \sum_{\lambda\in\{0\}\cup\Lambda_J} \phi_\lambda(x)\phi_\lambda(x^{'})$ with $\Lambda_J = \{(j,k),\ j \in \{0,\cdots,J-1\},\ k \in \{0,\cdots,2^j-1\}\}$. Let $\mathcal{M}_{\overline{J}} = \{J,\ 0 \leq J \leq 7\}$ and for every J in $\mathcal{M}_{\overline{J}}$, $w_J = 2\left(\ln(J+1) + \ln(\pi/\sqrt{6})\right)$.
We consider $\Phi^{(1)}_\alpha$ the multiple testing procedure with the collection of projection kernels $\{K_J,\ J \in \mathcal{M}_{\overline{J}}\}$. Second, for $\mathcal{L} = \{1,2,\cdots,6\}$ we take $\{h_l, l \in \mathcal{L}\} = \{1/24, 1/16, 1/12, 1/8, 1/4, 1/2\}$, let $K_l(x,y) = \frac{1}{h_l}k\left(\frac{x-y}{h_l}\right)$ with $k(u) = (2\pi)^{-1/2}\exp\left(-u^2/2\right)$. Then taking $w_l = 1/|\mathcal{L}| = 1/6$, we consider $\Phi^{(2)}_\alpha$ the multiple testing procedure denoted by G, with the collection of Gaussian kernels $\{K_l, l \in \mathcal{L}\}$. At last, we are interested in the collection of both projection and Gaussian kernels. We define $\Phi^{(3)}_\alpha$ denoted by PG, the multiple testing procedure with the collection of kernels $\{K_p,\ p \in \mathcal{P} = \mathcal{M}_{\overline{J}} \cup \mathcal{L}\}$. For $p \in \mathcal{M}_{\overline{J}}$ we take $w_p = \ln(J+1) + \ln(\pi/\sqrt{6})$ and for $p \in \mathcal{L}$ we take $w_p = 1/12$.
For each observation $X = (X_1,\cdots,X_n)$ we have to estimate $u^{(X)}_\alpha$ and $q^{(X)}_{m,1-u^{(X)}_\alpha e^{-w_m}}$ by the Monte Carlo method. Precisely, we generate 400000 samples of $\left\{\epsilon^b\right\}^{400000}_{b=1}$ and $\left\{\epsilon'^b\right\}^{400000}_{b'=1}$, in which we use one half to approximate the conditional probability and other half is used

	1st error	CI
P	0.0504	$[0.033, 0.068]$
G	0.0506	$[0.032, 0.068]$
PG	0.0498	$[0.032, 0.0657]$

Table 1: The probabilities of first kind error of test and their confidence interval (CI) with confidence level 99%.

to estimate the distribution of each $V_{K_m}^{(0)}$. $u_\alpha^{(X)}$ is approximated by taking u in a regular grid of $[0, 1]$ with bandwidth 2^{-16} and choosing the approximation of $u_\alpha^{(X)}$ as the largest value of the grid such that the estimated conditional probabilities are less than α.
We realize 5000 simulations of X. For each simulation, we determine the conclusions of the tests P, G and PG where the critical values are approximated by the Monte Carlo method. The probabilities of first kind error of the tests are estimated by the rejections for these tests divided by 5000 in the Table 1.
We then study the probabilities of rejection for each test for several alternatives including $f_{1,a,\epsilon}(x) = \epsilon \mathbb{1}_{[0,a)}(x) - \epsilon \mathbb{1}_{[a,2a)}(x)$, with $0 < \epsilon \leq 1$ and $0 < a < 1$; $f_{2,\tau}(x) = \tau \sum_j \frac{h_j}{2} (1 + \operatorname{sgn}(x - p_j))$, with $\tau > 0$, and $h_j \in \mathcal{Z},\ 0 < p_j < 1\ \forall j$; $f_{3,c}(x) = c \cos(10\pi x)$, with $c > 0$ and the last $f_{4,\varrho,j}(x) = \varrho \cos(2\pi j x)$, with $\varrho \geq 0$ and $j \in \mathbb{N} \setminus 0$ which we aim to compare our results with the results of Eubank and LaRiccia (1993) as

(a, ϵ)	$(1/4, 0.7)$		$(1/4, 0.9)$		$(1/4, 1)$		$(1/8, 1)$	
	$\hat{p}$	CI	$\hat{p}$	CI	$\hat{p}$	CI	$\hat{p}$	CI
P	0.876	$[0.849, 0.903]$	0.986	$[0.976, 0.996]$	0.996	$[0.990, 1.001]$	0.699	$[0.662, 0.736]$
G	0.831	$[0.801, 0.861]$	0.977	$[0.965, 0.989]$	0.992	$[0.985, 0.999]$	0.635	$[0.596, 0.674]$
PG	0.884	$[0.858, 0.910]$	0.984	$[0.973, 0.994]$	0.996	$[0.991, 1.001]$	0.690	$[0.652, 0.727]$

Table 2: The power of the test for the alternative $f_{1,a,\epsilon}$ corresponding to $(a, \epsilon) = (1/4, 0.7),\ (1/4, 0.9),\ (1/4, 1), (1/8, 1)$ and theirs CIs with confidence level 99%.

τ	0.05		0.1		0.5	
	$\hat{p}$	CI	$\hat{p}$	CI	$\hat{p}$	CI
P	0.218	$[0.177, 0.243]$	0.654	$[0.615, 0.693]$	1	*
G	0.208	$[0.175, 0.241]$	0.668	$[0.629, 0.704]$	1	*
PG	0.210	$[0.177, 0.243]$	0.678	$[0.639, 0.716]$	1	*

Table 3: The power of the test for the alternative $f_{2,\tau}$ corresponding to $\tau = 1, 2, 3$ and their CIs with confidence level 99%.

For each alternative f, we realize 1000 simulations of X. For each simulation, we determine conclusions of the tests P, G and PG, where the critical values of our tests are still approximated by the Monte Carlo method. The powers of the tests are estimated by the number of rejections divided by 1000, in the Table 2, 3, 4 and 5.

In the all cases, the three tests P, G and PG are powerful. In the three alternatives $f_{1,a,\epsilon}$, $f_{2,\tau}$ and $f_{3,c}$, the test PG is more powerful than P and G tests. Our conclusion is that the test PG is a good choice for practice. Comparing our estimated powers with the two tests denoted by EL1 corresponding to T_{nm} and EL2 corresponding to $T_{n\lambda}$ in Eubank

c	1		2		3	
	$\hat{p}$	ICI	$\hat{p}$	CI	$\hat{p}$	CI
P	0.35	$[0.311, 0.389]$	0.90	$[0.876, 0.924]$	0.98	$[0.969, 0.991]$
G	0.56	$[0.519, 0.600]$	0.98	$[0.967, 0.991]$	1	*
PG	0.34	$[0.301, 0.379]$	0.89	$[0.864, 0.915]$	1	*

Table 4: The power of the test for the alternative $f_{3,c}$ corresponding to $c = 1, 2, 3$ and their CIs with confidence level 99%.

		$\varrho = 0$		$\varrho = 0.5$		$\varrho = 1$		$\varrho = 1.5$	
		$\hat{p}$	CI	$\hat{p}$	CI	$\hat{p}$	CI	$\hat{p}$	CI
$j = 1$	P	0.049	$[0.031, 0.066]$	0.606	$[0.566, 0.645]$	1		1	
	G	0.048	$[0.031, 0.065]$	0.459	$[0.418, 0.499]$	0.99	$[0.982, 0.998]$	1	*
	PG	0.048	$[0.031, 0.065]$	0.441	$[0.401, 0.481]$	0.99	$[0.982, 0.998]$	1	*
	EL1	0.074	$[0.053, 0.095]$	0.837	$[0.807, 0.867]$	1	*	1	*
	EL2	0.062	$[0.042, 0.082]$	0.805	$[0.773, 0.837]$	1	*	1	*
$j = 3$	P	0.053	$[0.035, 0.071]$	0.224	$[0.190, 0.258]$	0.905	$[0.881, 0.928]$	1	*
	G	0.053	$[0.035, 0.071]$	0.224	$[0.190, 0.258]$	0.922	$[0.900, 0.944]$	1	*
	PG	0.049	$[0.031, 0.066]$	0.630	$[0.591, 0.669]$	1	*	1	*
	EL1	0.069	$[0.048, 0.089]$	0.718	$[0.681, 0.755]$	1	*	1	*
	EL2	0.058	$[0.039, 0.077]$	0.693	$[0.655, 0.731]$	1	*	1	*
$j = 6$	P	0.043	$[0.026, 0.060]$	0.134	$[0.106, 0.162]$	0.696	$[0.658, 0.733]$	0.990	$[0.982, 0.998]$
	G	0.044	$[0.027, 0.061]$	0.146	$[0.117, 0.174]$	0.741	$[0.705, 0.777]$	0.995	$[0.989, 1]$
	PG	0.045	$[0.028, 0.062]$	0.134	$[0.106, 0.162]$	0.700	$[0.663, 0.737]$	0.996	$[0.990, 1]$
	EL1	0.076	$[0.054, 0.098]$	0.134	$[0.106, 0.162]$	0.428	$[0.388, 0.468]$	0.979	$[0.967, 0.991]$
	EL2	0.056	$[0.037, 0.075]$	0.107	$[0.082, 0.132]$	0.368	$[0.328, 0.407]$	0.961	$[0.945, 0.977]$

Table 5: The power of the test for the alternative $f_{4,\varrho,j}$ corresponding to $\varrho = 0, 0.5, 1, 1.5,\ j = 1, 2, 3$ and their CIs with confidence level 99%.

and LaRiccia (1993) showed in the Table 5. We realize that our results are quite stable. Precisely, our results look not as good as the results in this paper in the case of $j = 1$ and $j = 3$, however, in the case of $j = 6$, the proportion of rejections in 1000 samples for various choices of ϱ in our results are more powerful than old results of this paper.

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