Statistical testing of the covariance matrix rank in multidimensional neuronal models

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Résumé. Le but de ce travail est de développer une procédure de test qui détermine le rang du bruit dans un processus stochastique multidimensionnel à partir d'observations discrètes de ce processus sur un intervalle de temps fixe $[0, T]$ échantillonné avec un pas de temps Δ. Nous utilisons l'approche de perturbation aléatoire, utilisée pour l'estimation du rang de matrices non aléatoires, dans le cas d'un processus de diffusion stochastique. Nous menons une étude de simulation sur des modèles stochastiques multidimensionnels de l'activité neuronale: le modèle FitzHugh-Nagumo et une approximation stochastique du processus de Hawkes. Notre objectif principal est de contrôler le taux de perturbation, qui garantit des statistiques non dégénérées utilisées dans le test, et d'étudier son influence sur la précision du test pour une taille de pas fixe Δ.

Mots-clés. Tests statistiques, diffusions hypoelliptiques, modèle neuronal FitzHugh-Nagumo, statistique computationnelle

Abstract. The aim of this work is to develop a testing procedure which determines the rank of the noise in a multidimensional stochastic process from discrete observations of this process on a fixed time interval $[0, T]$ sampled with a time step Δ. We use the random perturbation approach, used for non-random matrix rank estimation, to a stochastic diffusion process. We conduct a simulation study on multidimensional stochastic models of neuronal activity: FitzHugh-Nagumo model and a stochastic approximation of the Hawkes process. Our primary goal is to control the perturbation rate, which ensures a non-degenerate statistics used in the test, and study its influence on the test accuracy for a fixed step size Δ.

Keywords. Statistical tests, hypoelliptic diffusions, neuronal FitzHugh-Nagumo model, computational statistics

1 Introduction

Stochastic diffusions became a classical tool for describing a neuronal activity, either of a one single neuron (Ditlevsen and Samson, 2012, Höpfner et al., 2016, Leon and Samson, 2017), or a large network of neurons (Ditlevsen and Löcherbach, 2017, Ableidinger et al., 2017). However, the techniques which would allow us to establish a rigorous link between a specific model and available neurophysiological data is often missing.

The open question is the source of stochasticity in spiking activity. One point of view is that both the membrane and the ion channels of the neuron cell are affected by noise. Another position is that only the ion channels have a stochastic behaviour and that their concentration in cell explicitly defines the membrane potential. The question is then how to test both hypotheses with extracellular recordings of the membrane potential. For network-scale neuronal models, the estimation of the noise rank is equivalent to estimating a number of populations of different types of neurons in the network.

The question boils down to a problem of a covariance matrix rank estimation and constructing a statistical test of the rank. Our aim is to challenge this problem with the help of numerical approximation methods for stochastic diffusions and properties of matrix determinants, following works of Jacod et al. (2008), Jacod and Podolskij (2013).

2 Model

Consider a d-dimensional continuous Itô semimartingale X_t, given on some filtered probability space $(\Omega, \mathcal{F}, P)$, which is observed at equidistant times Δ over a time interval $[0, T]$. In a vector form it can be written as

$$dX_t = A_t dt + B_t dW_t, \tag{1}$$

where W_t is a standard q-dimensional Brownian motion, A_t is a d-dimensional drift process, B_t is a $R^{d\times q}$-valued volatility process, continuous in time. We further assume that the solution of (1) defined on $(\Omega, \mathcal{F}, P)$ can be represented in the following form:

$$\begin{aligned}
X_t &= X_0 + \int_0^t A_s ds + \int_0^t B_s dW_s \\
B_t &= B_0 + \int_0^t b_s ds + \int_0^t v_s dW_s \\
A_t &= A_0 + \int_0^t b'_s ds + \int_0^t v'_s dW_s \\
v_t &= v_0 + \int_0^t b''_s ds + \int_0^t v''_s dW_s,
\end{aligned}$$

where b'_t is R^d-valued, b_t and v'_t are $R^{d\times q}$-valued, v_t and b''_t are $R^{d\times q\times q}$-valued, and v''_s is $R^{d\times q\times q\times q}$-valued, all those processes are adapted. Finally, the processes b_t, v'_t, v''_t are càdlàg and the processes b'_t, b''_t are locally bounded.

Our goal is to determine the rank r_0 of the matrix BB^T and construct a statistical test which allows to test the null hypothesis $r = r_0$ against the alternative $r \neq r_0$ from discrete observations of the process $(X_{i\Delta}, i = 1 \dots N)$.

3 Statistical test

To begin with, let us recall some results from matrix algebra. Let Σ_1 be an unknown $d \times d$ matrix of unknown rank r_0. We assume that it is hard to compute the rank r_0 straightforwardly, but assume that there is a way to compute $\det(\Sigma_1 + h\Sigma_2)$ for some $h \ll 1$ and for some known $d \times d$ matrix Σ_2. Recall the following property of the matrix determinant. In 1×1-dimensional (scalar) case, $\det(\Sigma_1 + h\Sigma_2) = \det \Sigma_1 + h \det \Sigma_1$. In 2×2-dimensional case it is not that simple anymore, but we still can express $\det(\Sigma_1 + h\Sigma_2)$ as follows:

$$\det \Sigma_1 + h^2 \det \Sigma_2 + h \left[\gamma_1^1(\Sigma_1, \Sigma_2) + \gamma_1^2(\Sigma_1, \Sigma_2) \right],$$

where $\gamma_1^j, j = 1, 2$ stands for a determinant of a matrix, whose j-th column is a j-th column of a matrix Σ_1, and the remaining — the corresponding column of a matrix Σ_2.

Let us denote by γ_r a sum of the determinants of all the matrices obtained with a permutation of that type, that is with r columns taken from matrix Σ_1, and remaining $d - r$ — from matrix Σ_2. Indexes are preserved: i-th column of the resulting matrix is either i-th column of Σ_1 or i-th column of Σ_2. Then for $d \times d$-dimensional matrices, the determinant can be computed as follows:

$$\det(\Sigma_1 + h\Sigma_2) = \det \Sigma_1 + h\gamma_{d-1}(\Sigma_1, \Sigma_2) + \cdots + h^d \det \Sigma_2,$$

Formal proof of this result can be found in Jacod and Podolskii (2013) (Lemma 6.1).

Let us now assume that the rank of the matrix Σ_1 is equal to $r_0 = d - 1$, and that $h \ll 1$. Then the term $\det \Sigma_1$ vanishes, and the most important term, determining the behaviour of $\det(\Sigma_1 + h\Sigma_2)$ is $h\gamma_{d-1}(\Sigma_1, \Sigma_2)$. If $r_0 = d - 2$, then $h\gamma_{d-1}(\Sigma_1, \Sigma_2)$ vanishes as well, and so on. Finally, we have:

$$\det(\Sigma_1 + h\Sigma_2) = h^{d-r_0} \gamma_{r_0}(\Sigma_1, \Sigma_2) + \mathcal{O}(\Delta^{d-r_0+1}).$$

This naturally leads to the following result, which is the core of the estimating and testing procedure:

$$\frac{\det\left(\Sigma_1 + 2h\tilde{\Sigma}_2\right)}{\det\left(\Sigma_1 + h\tilde{\Sigma}_2\right)} \to 2^{d-r_0} \text{ as } h \to 0$$

Then, for h small emough, r_0 can be approximated as follows:

$$r_0 \approx d - \frac{\log \frac{\det\left(\Sigma_1 + 2h\tilde{\Sigma}_2\right)}{\det\left(\Sigma_1 + h\tilde{\Sigma}_2\right)}}{\log 2}. \qquad (2)$$

We use this property to study the empirical variance of the process (1), computed from the observations $(X_{i\Delta}, i = 1, \dots, N)$ on the fixed time interval. We introduce two new processes with 2 different orders of perturbation, namely, for $k = 1, 2$:

$$\tilde{X}_t^k = X_t + \tilde{\Sigma}\sqrt{k\Delta}\tilde{W}_t, \qquad (3)$$

where $\tilde{\Sigma}$ is an arbitrary chosen $d \times q$ non-random matrix of full rank, and $\tilde{W}_t$ is a q-dimensional Brownian motion. Key statistics S_t^1 and S_t^2 which will be plugged in formula (2) are defined as:

$$S_t^k = 2d\Delta \sum_{i=0}^{[t/2d\Delta]-1} s_i^k, \; k = 1, 2,$$

where

$$s_i^k = \det\left[\frac{\tilde{X}^{(k)}_{(2id+k):(2id+kd)} - \tilde{X}^{(k)}_{2id:(2id+kd-k)}}{\sqrt{k\Delta}}\right]^2.$$

Here by each matrix $\tilde{X}^{(k)}_{(2id+k):(2id+kd)} - \tilde{X}^{(k)}_{2id:(2id+kd-k)}$ we mean d successive increments of the process, taken with step $k\Delta$ and written column-by-column, so that in the end we obtain at most $\left[\frac{N}{2d}\right] - 1$ matrices of dimension $d \times d$. Also note that the increments are not overlapping, so that each i-th matrix is independent of its neighbors. The estimator of the rank BB^T is then defined as

$$\widehat{R}(T, \Delta) = d - \frac{\log(S_T^2/S_T^1)}{\log 2},$$

and its variance is computed as

$$V(T, \Delta) = Var\left[\widehat{R}(T, \Delta)\right] = \frac{\left(\frac{E[S_T^1]}{E[S_T^2]}\right)^2 Var[S_T^2] - 2\frac{E[S_T^1]}{E[S_T^2]}Cov[S_T^1 S_T^2] + Var[S_T^1]}{(E[S_T^1]\log 2)^2}.$$

Then we can use the following result for constructing the test (see Corollary 3.6 in Jacod and Podolskij (2013)):

$$\frac{\widehat{R}(T, \Delta) - r_0}{\sqrt{\Delta V(T, \Delta)}} \xrightarrow{\mathcal{L}} \zeta,$$

where $\zeta \sim \mathcal{N}(0, 1)$.

The null hypothesis "rank of the matrix BB^T equals to r" is then rejected if the computed value of the estimator belongs to the following critical (rejection) region:

$$\mathcal{C}(\alpha)_T = \left\{\omega : \left|\widehat{R}(T, \Delta) - r\right| > z_\alpha \sqrt{\Delta V(T, \Delta)}\right\},$$

where z_α is the symmetric α-quantile of $\mathcal{N}(0, 1)$.

4 Simulation study

We test the numerical performance of this estimator on simulated diffusion paths of the neuronal 2-dimensional FitzHugh-Nagumo model (both with full-rank and degenerate diffusion matrix) and the stochastic approximation of the mean-field limit of the Hawkes process with arbitrary number of memory variables. Naturally the rate of perturbation defined by a matrix $\tilde{\Sigma}$ influences the accuracy of the test, especially when the step size Δ is not sufficiently small. We aim to study and justify the optimal choice of the perturbation rate for models of different structure and explain, how the testing procedure can be adjusted.

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